

Galilean Exotic Planar Supersymmetries and Nonrelativistic Supersymmetric Wave Equations

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Abstract

We describe the general class of N -extended $D = (2 + 1)$ Galilean supersymmetries obtained, respectively, from the N -extended $D = 3$ Poincaré superalgebras with maximal sets of central charges. We confirm the consistency of supersymmetry with the presence of the ‘exotic’ second central charge θ . We show further how to introduce a $N = 2$ Galilean superfield equation describing nonrelativistic spin 0 and spin $\frac{1}{2}$ free particles.

1 Introduction

The Galilean invariance is the fundamental space-time symmetry of nonrelativistic systems. The Galilean projective representations (see e.g. [1]), used in the description of quantum d -dimensional nonrelativistic systems, are generated by a central extension of the $D = (d + 1)$ Galilei algebra called also Bargmann or quantum Galilei algebra, with mass m as a central charge. For example, for $d = 3$ the Galilei algebra takes the following form ($r, s, t = 1, 2, 3$; we assume that the generators are Hermitean):

$$\begin{aligned}
[J_r, J_s] &= i \epsilon_{rst} J_t, \\
[J_r, K_s] &= i \epsilon_{rst} K_t, \\
[K_r, K_s] &= 0, \quad [P_r, P_s] = 0, \\
[J_r, P_s] &= i \epsilon_{rst} P_t, \\
[H, J_s] &= 0, \quad [H, P_s] = 0, \\
[H, K_r] &= i P_r, \\
[P_r, K_s] &= i \delta_{rs} m,
\end{aligned} \tag{1}$$

where J_r describe $O(3)$ rotations, K_r - Galilean boosts, P_r - momenta and H is the energy operator. It is known that the Galilei algebra (1) can be obtained from the $D = 4$ Poincaré algebra by the relativistic contraction $c \rightarrow \infty$ ([2]).

The (2+1) dimensional Galilean algebra is a special case as, exceptionally, it allows for the existence of a second central charge θ . This algebra, with central charges m and θ , also called ‘exotic’, takes the following form (see e.g. [3]-[5]) ($i, j = 1, 2$)

$$\begin{aligned}
[J, K_i] &= i \epsilon_{ij} K_j, \\
[K_i, K_j] &= i \epsilon_{ij} \theta, \quad [P_i, P_j] = 0, \\
[J, P_i] &= i \epsilon_{ij} P_j, \quad [H, J] = [H, P_i] = 0, \\
[H, K_i] &= i P_i, \\
[P_i, K_j] &= i \delta_{ij} m,
\end{aligned} \tag{2}$$

with the generator J describing the $O(2)$ rotations. In [6] it was shown that the second central charge θ can be reinterpreted as introducing noncommutativity in the two-dimensional space

$$[\hat{x}_i, \hat{x}_j] = -i \frac{\theta}{m^2} \epsilon_{ij}. \tag{3}$$

The distinguished role of planar Galilean algebra follows from the covariance of the relations (3) under space rotations. For $d > 2$ (d - number of space dimensions) it is not possible to have noncommutativity relation for space coordinates which are covariant under classical $O(d)$ rotations what corresponds to the property that the respective Galilean algebras do not allow for the existence of the second central charge.

The first attempt at obtaining supersymmetric extensions of the $D = (3 + 1)$ Galilean algebra (1) was made by Puzalowski ([7]) who considered the $N = 1$ and $N = 2$ cases. The supersymmetric extension of the $D = (2 + 1)$ Galilean symmetries were considered for $N = 1$ and $N = 2$ in [8]-[11]. We should observe that the Galilean symmetries were extended to the so-called Schrödinger symmetries of free quantum mechanical systems (free nonrelativistic particle ([12]) and harmonic oscillator ([13])) by adding two additional generators: D (dilatations) and K (extensions, corresponding to the conformal time transformations). The Galilean superalgebra was then obtained as a subalgebra of supersymmetrically extended Schrödinger algebra ([9], [14]-[16]). However, the N extended supersymmetrization of the $D = (2 + 1)$ Galilean algebras have, so far, not been described in its general form; in particular, the $D = (2 + 1)$ Schrödinger super-algebra has never been written down in the presence of the second central charge θ .

The aim of this paper is twofold:

- To describe the new class of N -extended (N even) $D = (2+1)$ Galilean superalgebras, which we obtain by the nonrelativistic contraction $c \rightarrow \infty$ (c - velocity of light) of the corresponding $D = 3$ relativistic Poincaré superalgebras with maximal sets of central charges. We obtain $\frac{N(N-1)}{2}$ Galilean central charges - *i.e.* the same number as in the relativistic case¹. In Sect. 2. we show that the contraction of the central charges sector, which preserves their number, requires a suitable rescaling to obtain finite results in the $c \rightarrow \infty$ limit. We show further that for $D = (2 + 1)$ the two-parameter (m, θ) central extension (see (2)) is consistent with supersymmetry.

Note that an alternative split of supercharges into two sectors with different rescalings, by \sqrt{c} and $\frac{1}{\sqrt{c}}$ factors, was earlier considered for the $D = 10$ $N = 2$ SUSY in [17]. We shall adapt the method of [17] to $D = 3$ and compare the results with our contraction scheme.

- To present the realization of the $N = 2$ $D = (2 + 1)$ Galilean superal-

¹By central charges we denote here the Abelian generators which commute with supercharges and Galilean generators. In general they transform as tensors under the internal symmetry group.

gebra describing the supersymmetric nonrelativistic particle multiplet with spin 0 and $\frac{1}{2}$. In particular, we obtain the Levy-Leblond equations for nonrelativistic spin $\frac{1}{2}$ fields ([18]). The model can easily be made invariant under the exotic planar Galilean supersymmetry with the central charge $\theta \neq 0$.

Let us add that recently the Galilean supersymmetries have been applied to the light-cone description of superstrings ([19]), nonrelativistic super-p-branes ([20, 17]) and D-branes ([21]). We conjecture that our $D = (2 + 1)$ Galilean supersymmetries with central charge $\theta \neq 0$ can find application in the description of nonrelativistic supermembranes with noncommutative world volume geometry.

2 N - extended Galilean $D = (2 + 1)$ superalgebras as contraction limits

Let us recall that the N -extended $D = 3$ Poincaré superalgebra is given by $(\alpha, \beta = 1, 2; \mu = 0, 1, 2; A, B = 1..N)$:

$$\{Q_\alpha^A, Q_\beta^B\} = (\sigma_\mu)_{\alpha\beta} P^\mu \delta^{AB} + \epsilon_{\alpha\beta} Z^{AB}, \quad (4)$$

where $P^\mu = \eta^{\mu\nu} P_\nu$ ($\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$), $Z^{AB} = -Z^{BA}$ are $\frac{N(N-1)}{2}$ real central charges and $\sigma_\mu = (\sigma_i = \gamma_0 \gamma_i, \sigma_0 = \gamma_0^2 = -1_2)$. We choose

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon. \quad (5)$$

The full superalgebra is described by $2N$ real supercharges Q_α^A , the $D = 3$ Poincaré algebra ($P_\mu = (P_0, P_i)$, $M_{\mu\nu} = (M_{12} = J, M_{i0} = N_i)$), $\frac{N(N-1)}{2}$ central charges Z^{AB} and the $O(N)$ generators $T^{AB} = -T^{BA}$ describing internal symmetries:

- i) $D = 3$ Poincaré algebra

$$\begin{aligned} [J, N_i] &= i \epsilon_{ij} N_j \\ [N_i, N_j] &= i \epsilon_{ij} J \\ [J, P_i] &= i \epsilon_{ij} P_j, \quad [J, P_0] = 0 \\ [P_0, N_i] &= i P_i, \quad [P_0, P_i] = [P_i, P_j] = 0, \\ [P_i, N_j] &= i \delta_{ij} P_0. \end{aligned} \quad (6)$$

- ii) Supercovariance relations for $2N$ real supercharges Q_α^A

$$\begin{aligned} [Q_\alpha^A, N_i] &= \frac{i}{2}(\sigma_i)_{\alpha\beta} Q^{\beta A}, & [Q_\alpha^A, P_i] &= 0, \\ [Q_\alpha^A, J] &= \frac{i}{2}\epsilon_{\alpha\beta} Q^{\beta A}, & [Q_\alpha^A, P_0] &= 0. \end{aligned} \quad (7)$$

- iii) The internal index A is rotated by the $O(N)$ generators T^{AB} , where

$$\begin{aligned} [T^{AB}, P^\mu] &= [T^{AB}, M^{\mu\nu}] = 0, \\ [T^{AB}, T^{CD}] &= i(\delta^{AC}T^{BD} - \delta^{AD}T^{BC} + \delta^{BD}T^{AC} - \delta^{BC}T^{AD}) \end{aligned} \quad (8)$$

and

$$[T^{AB}, Q_\alpha^C] = i(\tau^{AB})_D^C Q_\alpha^D,$$

where $(\tau^{ab})_c^d$ describe the vectorial $N \times N$ matrix representation of $O(N)$ and

$$[T^{AB}, Z^{CD}] = i(\delta^{AC}Z^{BD} - \delta^{AD}Z^{BC} + \delta^{BD}Z^{AC} - \delta^{BC}Z^{AD}). \quad (9)$$

- iv) Central charges Z^{AB} are Abelian and commute with the supercharges Q_α^A and with $(P_\mu, M_{\mu\nu})$. For a given choice $Z_{(0)}^{AB}$ of the central charges the internal unbroken symmetry is described by the generators $\tilde{T}^{AB} \in T^{AB}$ which satisfy the relation (c.p. (9))

$$[\tilde{T}^{AB}, Z_{(0)}^{CD}] = 0. \quad (10)$$

The nonrelativistic contraction of the $D = 3$ Poincaré algebra part of the super-algebra (see (6)) is obtained by the introduction of the Hamiltonian H in the following way:

$$P_0 = mc + \frac{1}{c}H. \quad (11)$$

If we now perform the rescaling

$$N_i = cL_i$$

and take the limit $c \rightarrow \infty$ then we obtain from (6) the relations (2) with $\theta = 0$ and K_i replaced by L_i . In order to get the algebra (2) (*i.e.* the exotic (2+1) - dimensional Galilean algebra) one should perform the following linear change of basis

$$K_i = L_i + \frac{\theta}{2m}\epsilon_{ij}P_j. \quad (12)$$

In the case of a simple $N = 1$ $D = (2 + 1)$ Galilean superalgebra the relation (4) takes the simple form

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}P_0 + (\sigma_i)_{\alpha\beta}P_i. \quad (13)$$

Next we introduce the rescaled supercharges

$$S_\alpha = \frac{1}{\sqrt{c}}Q_\alpha \quad (14)$$

and using (13) find that in the $c \rightarrow \infty$ limit

$$\{S_\alpha, S_\beta\} = \delta_{\alpha\beta}m. \quad (15)$$

Further, from (7) and (14) we get the following completion of the $N = 1$ $D = 3$ Galilean super-algebra:

$$\begin{aligned} [K_i, S_\alpha] &= 0 \\ [J, S_\alpha] &= -\frac{i}{2}\epsilon_{\alpha\beta}S^\beta. \end{aligned} \quad (16)$$

Next we derive the N -extended $D = (2 + 1)$ -Galilean superalgebra for $N = 2k$ ($k = 1, 2, \dots$). We start from the superalgebra (4) where $A, B = 1, 2, \dots, 2k$. We define (see also [22])

$$Q_\alpha^{\pm a} = Q_\alpha^a \pm \epsilon_{\alpha\beta}Q_\beta^{k+a}, \quad (17)$$

where in the formula (17) we consider $a = 1, \dots, k$. Using (4) and (17) we get ($a, b = 1, \dots, k$)

$$\{Q_\alpha^{+a}, Q_\beta^{+b}\} = 2\delta_{\alpha\beta}P_0\delta^{ab} + \epsilon_{\alpha\beta}(Z^{ab} + Z^{\tilde{a}\tilde{b}}) + \delta_{\alpha\beta}(Y^{ab} - \tilde{Y}^{ab}), \quad (18)$$

$$\{Q_\alpha^{+a}, Q_\beta^{-b}\} = 2(\sigma_i P_i)_{\alpha\beta}\delta^{ab} + \epsilon_{\alpha\beta}(Z^{ab} - Z^{\tilde{a}\tilde{b}}) + \delta_{\alpha\beta}(Y^{ab} + \tilde{Y}^{ab}), \quad (19)$$

$$\{Q_\alpha^{-a}, Q_\beta^{-b}\} = 2\delta_{\alpha\beta}P_0\delta^{ab} + \epsilon_{\alpha\beta}(Z^{ab} + Z^{\tilde{a}\tilde{b}}) - \delta_{\alpha\beta}(Y^{ab} - \tilde{Y}^{ab}), \quad (20)$$

where the $2k \times 2k$ matrix of central charges Z^{AB} is described by the four $k \times k$ matrices

$$\begin{aligned} Z^{ab}, \quad \tilde{Z}^{ab} &= Z^{a+k, b+k}, \\ Y^{ab} &= Z^{k+a, b}, \quad \tilde{Y}^{ab} = Z^{a, k+b}, \end{aligned} \quad (21)$$

satisfying the symmetry properties:

$$Z^{ab} = -Z^{ba}, \quad \tilde{Z}^{ab} = -\tilde{Z}^{ba} \quad (22)$$

$$Y^{ab} = -\tilde{Y}^{ba}.$$

Finally, one gets $(Y^{(ab)} = \frac{1}{2}(Y^{ab} + Y^{ba}), Y^{[ab]} = \frac{1}{2}(Y^{ab} - Y^{ba}))$

$$\{Q_\alpha^{+a}, Q_\beta^{+b}\} = 2\delta_{\alpha\beta}(P_0\delta^{ab} + Y^{(ab)}) + \epsilon_{\alpha\beta}(Z^{ab} + \tilde{Z}^{ab}) \quad (23)$$

$$\{Q_\alpha^{+a}, Q_\beta^{-b}\} = 2((\sigma_i P_i)_{\alpha\beta}\delta^{ab} + \delta_{\alpha\beta}Y^{[ab]}) + \epsilon_{\alpha\beta}(Z^{ab} - \tilde{Z}^{ab}) \quad (24)$$

$$\{Q_\alpha^{-a}, Q_\beta^{-b}\} = 2\delta_{\alpha\beta}(P_0\delta^{ab} - Y^{(ab)}) + \epsilon_{\alpha\beta}(Z^{ab} + \tilde{Z}^{ab}). \quad (25)$$

Before taking the nonrelativistic limit we rescale the supercharges as follows:

$$S_\alpha^a = \frac{1}{\sqrt{c}}Q_\alpha^{+a}, \quad R_\alpha^a = \sqrt{c}Q_\alpha^{-a}. \quad (26)$$

The limit $c \rightarrow \infty$ of the relations (23-25) exists if we assume that

$$\begin{aligned} Y^{(ab)} &= \delta^{ab}mc + \frac{\tilde{Y}^{(ab)}}{c}, \\ Z^{[ab]} + \tilde{Z}^{[ab]} &= \frac{\tilde{U}^{[ab]}}{c} \\ Z^{[ab]} - \tilde{Z}^{[ab]} &= U^{[ab]} \end{aligned} \quad (27)$$

with central charges $Y^{[ab]}$, $U^{[ab]}$, $\tilde{Y}^{(ab)}$ and $\tilde{U}^{(ab)}$ having finite $c \rightarrow \infty$ limits. Using (26) and (27) we see that in this limit

$$\begin{aligned} \{S_\alpha^a, S_\beta^b\} &= 4m\delta_{\alpha\beta}\delta^{ab}, \\ \{S_\alpha^a, R_\beta^b\} &= 2(\sigma_i P_i)_{\alpha\beta} + \delta_{\alpha\beta}Y^{[ab]} + \epsilon_{\alpha\beta}U^{[ab]} \\ \{R_\alpha^a, R_\beta^b\} &= 2\delta_{\alpha\beta}(H\delta^{ab} - \tilde{Y}^{(ab)}) + \epsilon_{\alpha\beta}\tilde{U}^{[ab]}. \end{aligned} \quad (28)$$

We see that the N -extended $(2+1)$ -dimensional algebra ($N = 2k$) (28) contains the following set of central charges, with their number given in the second row,

$$\begin{array}{cccc} U^{[ab]}, & Y^{[ab]}, & \tilde{U}^{[ab]}, & \tilde{Y}^{(ab)} \\ \frac{k(k-1)}{2}, & \frac{k(k-1)}{2}, & \frac{k(k-1)}{2}, & \frac{k(k+1)}{2}, \end{array} \quad (29)$$

As $\frac{k(k+1)}{2} + 3\frac{k(k-1)}{2} = \frac{N(N-1)}{2}$ ($N = 2k$), the maximal number of central charges for the $D = 3$ Poincaré and $D = (2+1)$ Galilean symmetry is the same.

The supercovariance relations (7) take the following form

$$[N_i, Q_\alpha^{\pm a}] = -\frac{i}{2}(\sigma_i)_{\alpha\beta}Q_\beta^{\mp a} \quad (30)$$

and after rescalings (17) and (26) one gets, in the $c \rightarrow \infty$ limit

$$[L_i, S_\alpha^a] = 0, \quad [L_i, R_\alpha^a] = -\frac{i}{2}(\sigma_i)_{\alpha\beta} S_\beta^a \quad (31)$$

or, after performing the shift (12)

$$[K_i, S_\alpha^a] = 0, \quad [K_i, R_\alpha^a] = -\frac{i}{2}(\sigma_i)_{\alpha\beta} S_\beta^a. \quad (32)$$

Further, from (7) we get

$$[J, S_\alpha^a] = -\frac{i}{2} \epsilon_{\alpha\beta} S_\beta^a, \quad [J, R_\alpha^a] = -\frac{i}{2} \epsilon_{\alpha\beta} R_\beta^a. \quad (33)$$

It is easy to see that the relations (28) are covariant under the $O(\frac{N}{2}) = O(k)$ rotations, with indices a, b describing the k -dimensional vector indices. In comparison with (4) we see that the $O(N)$ covariance of the relations (4) has been reduced in the contraction procedure to the covariance with respect to the diagonal $O(k)$ symmetry obtained by constraining the $O(N)$ generators T^{AB} as follows:

$$T^{ab} = T^{k+a, k+b}, \quad T^{a, k+b} = 0. \quad (34)$$

In consequence, all the central charges (21) are the second rank $O(k)$ tensors and the Galilean supercharges (S_α^a, R_α^a) are the $O(k)$ vectors.

A special case corresponds to $k = 1$ ($N = 2$), when $Z^{AB} = \epsilon^{AB} Z$ defines the scalar central charge Z (from (22) we see that for $k = 1$, $Y = -\tilde{Y} = Z$). One gets

$$\begin{aligned} \{Q_\alpha^+, Q_\beta^+\} &= 2\delta_{\alpha\beta}(P_0 + Z) \\ \{Q_\alpha^+, Q_\beta^-\} &= 2(\sigma_i P_i)_{\alpha\beta} \\ \{Q_\alpha^-, Q_\beta^-\} &= 2\delta_{\alpha\beta}(P_0 - Z). \end{aligned} \quad (35)$$

Writing the first relation in (28) in the form $Z = mc + \frac{U}{c}$ and using (11) and (27) we get, when $c \rightarrow \infty$

$$\begin{aligned} \{S_\alpha, S_\beta\} &= 4m\delta_{\alpha\beta} \\ \{S_\alpha, R_\beta\} &= 2(\sigma_i P_i)_{\alpha\beta} \\ \{R_\alpha, R_\beta\} &= 2(H - U)\delta_{\alpha\beta}, \end{aligned} \quad (36)$$

where U plays the role of the correction to the Hamiltonian originating from the central charge Z . Further, from the relation (8) we get ($T^{AB} = \epsilon^{AB} T$; $A, B = 1, 2$)

$$[T, Q_\alpha^\pm] = \mp i Q_\alpha^\pm \quad (37)$$

and after contraction

$$[T, S_\alpha] = -i S_\alpha, \quad [T, R_\alpha] = i R_\alpha \quad (38)$$

In addition we get, using (12), (27) and the contraction limit

$$[K_i, S_\alpha] = 0, \quad [K_i, R_\alpha] = -\frac{i}{2}(\sigma_i)_{\alpha\beta} S_\beta \quad (39)$$

as well as

$$[J, S_\alpha] = -\frac{i}{2} \epsilon_{\alpha\beta} S_\beta, \quad [J, R_\alpha] = -\frac{i}{2} \epsilon_{\alpha\beta} R_\beta. \quad (40)$$

In our contraction scheme we employ the split (17) of the supercharges with two sectors undergoing different rescalings. If we express the relations (17) as ($a = 1, 2$, $\alpha, \beta = 1 \dots k$; $N=2k$)

$$Q_\alpha^{\pm a} = (\mathcal{P}^\pm)_{\alpha\beta}^{ab} Q_\beta^b, \quad (41)$$

then the $N \times N$ matrices \mathcal{P}^\pm do not describe the projection operators². For $k = 1$ ($N = 2$) by adapting to $D = 3$ the contraction scheme proposed in [17] we can, however, introduce an alternative split of the supercharges Q_α^a

$$\tilde{Q}_\alpha^{\pm a} = (P_\pm Q)_\alpha = Q_\alpha^a \pm \epsilon_{\alpha\beta} \epsilon^{ab} Q_\beta^b, \quad (42)$$

where P_\pm are the projection operators satisfying additional properties

$$P_\pm^T = P_\pm \quad (P_\pm)_{\alpha\beta}^{ab} = \epsilon_{\alpha\gamma} \epsilon^{ac} P_{\gamma\beta}^{cb}. \quad (43)$$

As for $N = 2$ we have $Z_{AB} = \epsilon_{AB} Z$ ($A, B = 1, 2$) we see from (4) that

$$\begin{aligned} \{\tilde{Q}_\alpha^{\pm a}, Q_\beta^{\pm b}\} &= 2(P_\pm)_{\alpha\beta}^{ab} (P_0 \pm Z), \\ \{Q_\alpha^{+a}, Q_\beta^{-b}\} &= 2(P_+)_{\alpha\gamma}^{ab} (\sigma_i P_i)_{\gamma\beta}. \end{aligned} \quad (44)$$

Introducing, in analogy with (26), the rescaling

$$\tilde{S}_\alpha^a = \frac{1}{\sqrt{c}} \tilde{Q}_\alpha^{+a}, \quad \tilde{R}_\alpha^a = \sqrt{c} \tilde{Q}_\alpha^{-a}, \quad (45)$$

and using the relations (11) and (27) we get in the limit $c \rightarrow \infty$ the following $N = 2$ $D = 3$ nonrelativistic Galilean superalgebra

²Because $\mathcal{P}^\pm = \begin{pmatrix} I_k & 0 \\ 0 & \epsilon I_k \end{pmatrix}$ we get

$$\mathcal{P}_\pm \mathcal{P}_\pm = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}; \quad \mathcal{P}_\pm \mathcal{P}_\mp = I_N.$$

$$\begin{aligned}
\{\tilde{S}_\alpha^a, \tilde{S}_\beta^b\} &= 4m(P_+)_{\alpha\beta}^{ab}, \\
\{\tilde{S}_\alpha^a, \tilde{R}_\beta^b\} &= 2(P_+)_{\alpha\beta}^{ab}(\sigma_i P_i)_{\gamma\beta}, \\
\{\tilde{R}_\alpha^a, \tilde{R}_\beta^b\} &= 2(P_-)_{\alpha\beta}^{ab}(H - U).
\end{aligned} \tag{46}$$

The $D = 10$ analogue of the superalgebras (44) and (46), under a simplifying assumption $U = -H$, has been proposed in [17]. It appears that the projections (42) and the Galilean superalgebra (46) can be used in supersymmetric D -brane models for the introduction of nonrelativistic kappa-symmetries which eliminate half of the fermionic degrees of freedom.

Finally we add that if $k > 1$ the definition (42) and the appearance of kappa transformations which eliminate half of the fermionic degrees of freedom remain valid in the presence of one nonvanishing central charge. In the presence of several relativistic central charges the split of supercharges into the parts undergoing \sqrt{c} and $\frac{1}{\sqrt{c}}$ rescalings is not unique, and the kappa transformations eliminate less than half of the fermionic variables.

3 Superfield wave equation with standard and exotic $N = 2$ Galilean supersymmetry

In this section we derive a super-Galilean covariant wave equation for our superfield $\Psi(t, x; \theta, \eta)$ where θ and η two real valued anticommuting spinors with components θ_α and η_β .

The proposed superfield equation is the following (see also [23]):

$$\nabla_\alpha \Psi = 0, \tag{47}$$

where ∇_α is a supercovariant derivative³.

Here, we will describe a derivation of (47) without the need of the Lagrangian.

In order to determine the explicit form of our supercovariant derivative we require, in addition to (47) that we have

$$[\nabla_\alpha, H] \Psi = 0, \quad [\nabla_\alpha, P_i] \Psi = 0, \tag{48}$$

$$\{\nabla_\alpha, S_\beta\} \Psi = 0, \quad \{\nabla_\alpha, R_\beta\} \Psi = 0, \tag{49}$$

and

$$[\nabla_\alpha, J] \Psi = 0, \quad [\nabla_\alpha, K_i] \Psi = 0. \tag{50}$$

³Note that the supercovariance of this superderivative holds only in a weak sense, *ie* as is clear from (56,57), it is valid only on the solutions of the wave equation (47)

In addition (47) ought to be the “square-root” of the Schrödinger equation [15]; *i.e.* we require that

$$\{\nabla_\alpha, \nabla_\alpha\} \Psi \sim (H - \frac{\vec{p}^2}{2m}) \Psi. \quad (51)$$

The super-derivative ∇_α , satisfying all the requirements (48-51) is given by

$$\nabla_\alpha = \frac{1}{2} (\sigma_i P_i)_{\alpha\beta} S_\beta - m R_\alpha. \quad (52)$$

To prove our claim we use the $N = 2$ super-Galilean algebra with a vanishing central charge U discussed in section 2.

First we note that the invariance of ∇_α with respect to space-time translations is evident. Furthermore,

1. from (36) we have

$$\{\nabla_\alpha, \nabla_\alpha\} = 4m^2 (H - \frac{\vec{P}^2}{2m}) \quad (53)$$

2. and

$$\{\nabla_\alpha, S_\beta\} = 0 \quad (54)$$

3. from (39) and (40)

$$[\nabla_\alpha, J] = \frac{1}{2} (\sigma_i)_{\alpha\beta} \nabla_\beta, \quad [\nabla_\alpha, K_i] = 0 \quad (55)$$

4. from (36)

$$\{\nabla_\alpha, R_\beta\} = 2m \delta_{\alpha\beta} (\frac{\vec{P}^2}{2m} - H) \quad (56)$$

and therefore, due to (47) and (53)

$$\{\nabla_\alpha, R_\beta\} \Psi = 0. \quad (57)$$

In order to express the wave equation (47) with ∇_α given by (52), in terms of derivatives with respect to the arguments of Ψ , we need a realisation of the $N = 2$ super-Galilei algebra in terms of differential operators. It is easily seen that such a realisation is given by ($U = 0$):

$$H = i\partial_t, \quad P_i = -i\partial_i, \quad (58)$$

$$K_i = tP_i - mx_i + \frac{\theta}{2m} \epsilon_{ij} P_j + \frac{i}{2} \eta_\alpha (\sigma_i)_{\alpha\beta} \frac{\partial}{\partial \theta_\beta}, \quad (59)$$

$$J = \epsilon_{ij}x_iP_j + \frac{i}{2}\epsilon_{\alpha\beta}\left(\theta_\alpha\frac{\partial}{\partial\theta_\beta} + \eta_\alpha\frac{\partial}{\partial\eta_\beta}\right), \quad (60)$$

$$S_\alpha = 2\frac{\partial}{\partial\theta_\alpha} + \frac{1}{2}(\sigma_iP_i)_{\alpha\beta}\eta_\beta + m\theta_\alpha, \quad (61)$$

$$R_\alpha = 2\frac{\partial}{\partial\eta_\alpha} + \frac{1}{2}(\sigma_iP_i)_{\alpha\beta}\theta_\beta + \frac{i}{2}\eta_\alpha\partial_t, \quad (62)$$

where the spinor derivatives are defined as left-derivatives and the x_i are commuting variables.

Note that the spinor part of K_i is necessary so that K_i has the correct expressions for its commutators with the supercharges S_α and R_α . This extra term leads also to a nontrivial behaviour of the spinor θ_α with respect to boosts *i.e.*:

$$[K_i, \theta_\alpha] = \frac{i}{2}(\sigma_i)_{\alpha\beta}\eta_\beta. \quad (63)$$

From (61) and (62) we can read off the transformation properties of our variables in superspace $Y \in (t, x, \theta, \eta)$ under infinitesimal supertranslations. Then with

$$\delta Y = [Y, \epsilon_\beta S_\beta + \rho_\beta R_\beta] \quad (64)$$

we obtain

$$\delta\eta_\alpha = 2\rho_\alpha, \quad \delta\theta_\alpha = 2\epsilon_\alpha, \quad (65)$$

$$\delta x_i = \frac{i}{2}(\epsilon_\beta(\sigma_i)_{\beta\gamma}\eta_\gamma + \rho_\beta(\sigma_i)_{\beta\gamma}\theta_\gamma), \quad (66)$$

$$\delta t = -\frac{i}{2}\rho_\beta\eta_\beta. \quad (67)$$

Finally, using (61) and (62) for the supercharges S_α , R_α we obtain the following decomposition of the super-covariant derivative ∇_α

$$\nabla_\alpha = D_\alpha + \frac{1}{4}\eta_\alpha(\vec{P}^2 - 2mi\partial_t), \quad (68)$$

where

$$D_\alpha = (\sigma_iP_i)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} - 2m\frac{\partial}{\partial\eta_\alpha}. \quad (69)$$

Thus the wave equation (47) is equivalent to the following pair of differential equations:

$$D_\alpha\Psi = 0, \quad (70)$$

with D_α given by (69) and

$$\left(i\partial_t - \frac{\vec{P}^2}{2m}\right)\Psi = 0. \quad (71)$$

As the superfield ψ has the expansion

$$\Psi(t, x; \theta_\alpha, \eta_\alpha) = \phi(t, x) + \theta_\alpha \psi_\alpha(t, x) + \eta_\alpha \chi_\alpha(t, x) + \dots \quad (72)$$

the equation (70) gives us the following equation for the spinor fields

$$(\sigma_i P_i)_{\alpha\beta} \psi_\beta(t, x) - 2m \chi_\alpha(t, x) = 0 \quad (73)$$

which, when combined with (71) gives us

$$i\partial_t \psi_\alpha = \frac{1}{2} (\sigma_i P_i)^2 \psi_\alpha = (\sigma_i P_i)_{\alpha\beta} \chi_\beta. \quad (74)$$

The set of Eqs.(73)-(74) provides the Levy-Leblond equations for the nonrelativistic spin 1/2 particles ([18]; see also [8]).

4 Conclusions

In this paper we have restricted ourselves to the case of the $D = 2 + 1$ nonrelativistic supersymmetries but the discussion of the $D = 3 + 1$ case can be performed in an analogous way. In particular, the basic equations (52), (70) and (73) can be extended to a three-dimensional space by introducing complex two-component spinors $\theta_\alpha, \eta_\alpha$ and three 2x2 complex Pauli matrices.

Note that in Section 3 we derived the superfield form of the nonrelativistic Levy-Leblond equations without postulating a classical action in the $N = 2$ $D = 3$ superspace $(x_i, t, \theta_\alpha, \eta_\alpha)$. Note also that our superfield equation (see (47) and (68)) does not depend on the exotic parameter θ .

In Section 2 we have presented the new general nonrelativistic contraction scheme for the N -extended $D = 3$ Poincaré algebra. Had we started from the most general $D = 4$ N-extended Poincaré algebra with $\frac{N(N-1)}{2}$ complex central charges ([24]), we would have obtained analogous results for the $D = 3+1$ nonrelativistic supersymmetric theory, with equal numbers of relativistic and nonrelativistic complex central charges.

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